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On Deficient Cubic Spline Interpolants

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We study cubic spline interpolation with less restrictive continuity requirements at the knots. We establish an interpolant which can interpolate at any point of the partition and also matches the area with certain mean over a greater partition length. A non-existence case has also been obtained.

1. DEFINITIONS

Let $0=x_0<x_1<x_2 \cdots <x_n=1$, $n>1$ be given points in $[0, 1]$. The set of points is called a partition and is denoted by Δ. The distance between these points, i.e., x_i-x_{i-1}, $i=1, 2, ..., n$, is denoted by h_i. In case these points are equidistant, we write $h_i=h$. Let $\pi_m[\mathrm{a}, \mathrm{b}]$ denote the set of all real algebraic polynomials of degree not greater than m defined on $[a, b]$. The class of deficient polynomial splines is defined as

$$S(m, \Delta)=\{s(x)\colon s(x)\in C^{m-2}[0, 1],\ s(x)\in \pi_m[x_{i-1}, x_i],\ i=1, ...\, n\}.$$

We take $g(x)$, defined on $[0, 1]$, with the property that $g(x+h)-g(x)$ is constant.

2. INTRODUCTION AND MAIN RESULTS

For obtaining the cubic spline for a function we need a condition for each partition on the spline depending on the function, e.g., interpolation and some boundary conditions in general. The deficient cubic spline is useful as it has less restrictive continuity requirements at the knots and this is compensated by taking another interpolatory condition or a more general condition like mean averaging (e.g., [2, 3, 5]). The following problem was investigated in [4].

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PROBLEM. *Let f be a* 1*-periodic locally integrable function with respect to a positive measure dg. Find* 1*-periodic* $s(x) \in S(3, \Delta)$ *satisfying the conditions*

$$\text{(i)}\quad f(\theta_i) = s(\theta_i), \qquad \text{(ii)}\quad \int_{x_{i-1}}^{x_i} (f-s)(x)\, dg = 0, \quad i = 1, ..., n, \tag{1}$$

where g is such that (ii) *does not reduce into* (i), *and* $\theta_i = x_{i-1} + mh$, $0 \leqslant m \leqslant 1$.

They established the existence of a unique $s(x)$ under either of the conditions.

$$g(x) = \text{constant} \begin{cases} \text{(i)}\ \ 0 \leqslant x < mh,\ 2/9 \leqslant m < 1/2 \\ \text{(ii)}\ \ mh \leqslant x < h,\ 1/2 \leqslant m \leqslant 7/9. \end{cases} \tag{2}$$

If we interpolate at the mid-point, that is, $m = 1/2$, we find from (2) that the matching of the area, with the mean, is obtained over half of the partition length. In case g is a single jump function having unit jump, the mean averaging condition reduces to interpolation at the point of the jump. Further, the practical choice of the points of interpolation are either knots or mid-points or both of the partition. We prove the following theorem which includes not only all the above cases but also allows us to interpolate at any point of the mesh. The area matching is achieved over a greater partition length.

THEOREM 1. *There exists a unique cubic spline* $s(x) \in S(3, \Delta)$ *satisfying the condition* (1) *for the locally integrable function f with respect to the positive measure dg, and having simple supports at the ends, precisely* $s'(1) = s'(0) = 0$, *provided* $g(x) =$ *constant either in* $(0, mh)$, $m \geqslant (3 - \sqrt{7})/4$ (< 0.09) *or in* (mh, h), $m \leqslant (1 + \sqrt{7})/4$ (> 0.91).

3. LEMMAS

We need the following lemmas for the proof of Theorem 1.

LEMMA 1. *Let dg be a positive measure, then*

$$K_1(m) = h^{-3} \int_0^h \{-2x^2 + h(3-2m)(x+mh)\}(x - mh)\, dg \neq 0,$$

if $g(x)$ *does not have just a single jump at* $x = mh$, $0 \leqslant m \leqslant 1$, *and g is constant either in* $(0, mh)$ *or in* (mh, h).

Proof (*cf.* [4, Lemma 1]). We see that the derivative of the integrand is $6x(h-x)$. Thus $x^2(3h-2x)$ is nondecreasing. From this we obtain that the integrand is negative in $0 \leqslant x < mh$, positive in $mh < x \leqslant h$, and 0 at mh. This proves the lemma.

LEMMA 2. *Let $C_n(c_1, c_2)$ be the following n dimensional tridiagonal matrix of real numbers*:

$$\begin{bmatrix} q & p & 0 & 0 & \cdots & 0 & 0 & c_1 \\ r & q & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & r & q & p & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_2 & 0 & 0 & 0 & \cdots & 0 & r & q \end{bmatrix}.$$

Then $C_n(0, 0)$ is nonsingular if $q^2 \geqslant 4pr$ except the case

(i) $q = 0$ *when* $q^2 = 4pr$ *or n is odd.*

For the case (i) *the matrix is singular. The result is the best possible in the sense that it breaks down if $q_2 < 4pr$.*

Proof. Let $D_n(c_1, c_2)$ denote the determinant of the matrix $C_n(c_1, c_2)$. We can see that

$$D_{n+1}(0, 0) = qD_n(0, 0) - prD_{n-1}(0, 0). \tag{3}$$

Equation (3) solves $D_n(0, 0) = (n+1)(q/2)^n$ if $q^2 = 4pr$ and $D_n(0, 0) = (\beta_1^{n+1} - \beta_2^{n+1})/(\beta_1 - \beta_2)$ if $q^2 \neq 4pr$, where $\beta_1 - \beta_2 = \sqrt{q^2 - 4pr}$ and $\beta_1 + \beta_2 = q$. If $q^2 > 4pr$, then $\beta_1 > |\beta_2|$ for $q > 0$ and $-\beta_2 > |\beta_1|$ for $q < 0$. This yields that $D_n(0, 0) \neq 0$. That the matrix is singular for the case (i) is direct. If $4pr > q^2 \geqslant 0$, then $D_n(0, 0) = (\sqrt{pr})^n \sin(n+1)\theta/\sin\theta$ where $0 < \theta < \pi$ and $\cos\theta = q/2\sqrt{pr}$. Thus, there is a discrete spectrum of permissible values of $q/\sqrt{pr}$ for which some $C_n(0, 0)$ are singular.

Remark. It can be seen by the cofactor expansion that

$$D_n(r, p) = (\beta_1^{n+1} - \beta_2^{n+1})/(\beta_1 - \beta_2) + (-1)^{n+1}(p^n + r^n).$$

It is interesting to observe that Theorem 2 for the case of uniform partition in [5] follows directly from this lemma.

For the sake of convenience, from here on we write $D_\tau(0, 0) = D_\tau$ and $D_0 = 1$, $D_1 = q$, $D_2 = q^2 - pr$, $D_3 = q(q^2 - 2pr)$.

LEMMA 3. *Let $C_n^{-1}(0,0)=[\hat{a}_{ij}]$. Then we have*

$$\hat{a}_{ij}=\begin{cases}(-r)^{i-j}D_{j-i}D_{n-i}/D_n, & i\geqslant j\\ (-p)^{j-i}D_{i-1}D_{n-j}/D_n, & i\leqslant j.\end{cases}$$

Proof. We first claim that

$$D_n=D_{n-i+1}D_{i-1}-pr\,D_{n-i}D_{i-2},\qquad n=3,4,...,2\leqslant i\leqslant n. \tag{4}$$

That (4) holds for $n=3$ and $n=4$ is direct. Using (4) for $n-1$ and n, we see from (3) that (4) holds for $n+1$ in place of n.

Now assume that $[\hat{a}_{ij}]\,C_n(0,0)=[b_{ij}]$. We have, for $1<i<n$, $D_nb_{ii}=-pr\,D_{i-2}\,D_{n-i}+q\,D_{i-1}\,D_{n-i}-rp\,D_{i-1}D_{n-i-1}=D_n$ and $b_{11}=b_{nn}=1$. For $i>j$,

$$\begin{aligned}D_nb_{ij}&=p(-r)^{i-j+1}D_{j-2}D_{n-i}+q(-r)^{i-j}D_{j-1}D_{n-i}\\&\quad+r(-r)^{i-j-1}D_jD_{n-i}\\&=r(-r)^{i-j-1}D_{n-i}[-D_j+D_j]=0.\end{aligned}$$

Similarly, for $i<j$, we get $D_nb_{ij}=0$.

Hence the lemma.

4. PROOF OF THEOREM 1

For $i=1,2\ldots$ n, we set

$$\int_{x_{i-1}}^{x_i} f\,dg=F_i \qquad\text{and}\qquad \int_0^h x^k(h-x)^l\,dg=h^{k+l}K(k,l),\quad k,l=0,1,2,3.$$

Let $s'(x_i)=M_i$, $i=0,1,...,n$. In view of Lemma 1, we find (cf. [4, (3.6)]) that the spline exists uniquely if the following linear system of equations has unique solutions.

$$p(m)M_{i+1}+q(m)M_i+r(m)M_{i-1}=u_i,\qquad 1\leqslant i<n, \tag{5}$$

where

$$\begin{aligned}p(m)&=h^{-1}m^2\,[K(3,0)-mK(2,0)],\\q(m)&=h^{-1}\,[(1-m)\,m^2\,K(0,0)+m^2\,(3-2m)K(1,0)\\&\qquad+(2m^2-2m-1)\{(m+1)K(2,0)-K(3,0)\}],\\r(m)&=h^{-1}(1-m)^2\,[-mK(0,0)+(2m+1)\,K(1,0)\\&\qquad-(m+2)\,K(2,0)+K(3,0)],\end{aligned}$$

and

$$u_i = h^{-2}\{(1-3m^2+2m^3)F_i + (3m^2-2m^3)F_{i+1} - f(\theta_i)K(0,0) + (3K(2,0)-2K(3,0))(f(\theta_i)-f(\theta_{i+1}))\}.$$

Now we show that the linear system of equations has unique solutions. Since by the boundary conditions $M_0 = M_n = 0$, the spline exists if $C_{n-1}(0,0)$ is nonsingular with $p = p(m)$, $r = r(m)$, and $q = q(m)$. In view of Lemma 2 it suffices to show that $q^2 \geqslant 4pr$. It can be seen that the integrands in $p(m)$, $q(m)$, and $r(m)$ are

$$p'(m,x) = h^{-4}m^2x^2(x-mh),$$

$$q'(m,x) = -h^{-4}\{x(1+2m-2m^2)(h-x)(x-mh) + h^2m(1-m)(x-mh)\},$$

and

$$r'(m,x) = h^{-4}\{(1-m)^2(h-x)^2(x-mh)\},$$

respectively. Taking the first term of $q'(m,x)$ as $q_1'(m,x)$ and the second term as $q_2'(m,x)$ and observing that

$$\int_{mh}^{h} p'(m,x)\,dg \int_{mh}^{h} r'(m,x)\,dg \leqslant h^{-4}m^2(1-m)^4\left(\int_{mh}^{h}(x-mh)\,dg\right)^2$$

we get

$$\left(\int_{mh}^{h} q_2'(m,x)\,dg\right)^2 \geqslant \mu \int_{mh}^{h} p'(m,x)\,dg \int_{mh}^{h} r'(m,x)\,dg,$$

if

$$1 \geqslant \mu(1-m)^2. \tag{6}$$

Further

$$2\int_{mh}^{h} q_1'(m,x)\,dg \int_{mh}^{h} q_2'(m,x)\,dg$$

$$\geqslant 2m^2h^{-5}(1-m)(1+2m-2m^2)\int_{mh}^{h}(x-mh)\,dg \int_{mh}^{h}(h-x)(x-mh)\,dg.$$

While

$$\int_{mh}^{h} p'(m,x)\,dg \int_{mh}^{h} r'(m,x)\,dg$$

$$\leqslant m^2h^{-5}(1-m)^3\int_{mh}^{h}(x-mh)\,dg \int_{mh}^{h}(h-x)(x-mh)\,dg.$$

Hence for a number μ, $1 \leqslant \mu < 4$,

$$2 \int_{mh}^{h} q_1'(m, x)\, dg \int_{mh}^{h} q_2'(m, x)\, dg \geqslant (4-\mu) \int_{mh}^{h} p'(m, x)\, dg \int_{mh}^{h} r'(m, x)\, dg$$

if

$$2(1+2m-2m^2) \geqslant (4-\mu)(1-m)^2. \tag{7}$$

Thus for constant $g(x)$, $x \in (0, mh)$, $q^2 \geqslant 4pr$ if (6) and (7) hold. These inequalities imply $1 \geqslant m \geqslant (3-\sqrt{7})/4$.

Similarly, we obtain that

$$\left(\int_0^{mh} q'(m, x)\, dg\right)^2 \geqslant 4 \int_0^{mh} p'(m, x)\, dg \int_0^{mh} r'(m, x)\, dg$$

is true if

$$0 \leqslant m \leqslant (1+\sqrt{7})/4.$$

This proves the theorem.

5. Error Estimate

We obtain the following error bounds for the spline of Theorem 1. We write $e(x) = s(x) - f(x)$.

THEOREM 2. *For $f \in C^1[0, 1]$, we have*

$$\max_i |e'(x_i)| \leqslant CC(m)\, \omega(f', h),$$

where $C = \|C_n^{-1}(0, 0)\|$, the norm is the row max norm, and $C(m)$ is

$$[(-3m^3+3m^2+m)\, K(0, 0) + (-4m^3+6m^2+1)\, K(1, 0) + (m^3+4m+3)\, K(2, 0) + (3m+2)\, K(3, 0)]/h.$$

It may be seen from the definition of $K(k, l)$ that the factor $C(m)$ is independent of h for the function $g(x)$ whose variation in $(0, h)$ is of order h.

Proof of Theorem 2. The linear system of equations (5) can be written as

$$[a_{ij}][M_i] = [u_i], \tag{8}$$

where $[a_{ij}]$ is the coefficient square matrix of order $n-1$ and $[M_i]$ and $[u_i]$ are single column matrices. We see from (8) that

$$[a_{ij}][e'(x_i)] = [u_i] - [a_{ij}][f'(x_i)].$$

Consequently,

$$\begin{aligned}\max_i |e'(x_i)| &\leqslant C\|[u_i] - [a_{ij}][f'(x_i)]\| \\ &= Ch^{-2}\max_i |m^2(3-2m)(F_{i+1}-F_i) + F_i - f(\theta_i)\,K(0,0) \\ &\quad + (3K(2,0)-2K(3,0))(f(\theta_i)-f(\theta_{i+1})) \\ &\quad - h^2\{r(m)f'(x_{i-1}) + q(m)f'(x_i) + p(m)f'(x_{i+1})\}|.\end{aligned}$$

By $\eta_i(x)$ we mean points in $[x_{i-1}, x]$; we write $\eta_i(x_i) = \eta_i$. Using the Taylor's formula, we obtain

$$\begin{aligned}F_{i+1} - F_i = \int_{x_i}^{x_{i+1}} f'(\eta_{i+1}(x))(x-x_i)\,dg + h\int_{x_{i-1}}^{x_i} f'(\eta_i(x))\,dg \\ - \int_{x_{i-1}}^{x_i} f'(\eta_i(x))(x-x_{i-1})\,dg,\end{aligned}$$

$$F_i - f(\theta_i)\,K(0,0) = \int_{x_{i-1}}^{x_i} f'(\eta_i(x))(x-x_{i-1})\,dg - mh\int_{x_{i-1}}^{x_i} f'(\eta_i(\theta_i))\,dg$$

and

$$f(\theta_i) - f(\theta_{i+1}) = h[mf'(\eta_i(\theta_i)) - f'(\eta_i) - mf'(\eta_{i+1}(\theta_{i+1}))].$$

Hence

$$\max_i |e'(x_i)| \leqslant CC(m)\,\omega(f', h).$$

6. Area Matching Case

In the following theorem, we show existence of the spline which bounds the same area as the function does and matches it at any desired point.

THEOREM 3. *There exists a unique spline considered in Theorem* 1 *such the area bounded by the spline and the function are the same if and only if* $m \neq 1/2$ *when n is odd.*

Proof. Taking $dg = dx$, we see that $K_1(m) > 0$ and from (5), we have

$$p(m) = m^2(3-4m)/12; \qquad r(m) = (1-m)^2\,(1-4m)/12$$

and

$$q(m) = (2m-1)(1+8m-8m^2)/12.$$

For $m \in [0, 1/4]$ and $[3/4, 1]$, $|q| > |p| + |r|$ and for the other values of m, p and r are of the opposite sign. Thus $q^2 \geqslant 4pr$ for $m \in [0, 1]$. $q = 0$ only for $m = 1/2$. In view of this the theorem follows from Lemma 2.

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